Markov process built in scale-similar multifractal energy cascades in turbulence

Iwao Hosokawa
2-24-6-101, Honmachi, Fuchu, Tokyo 183-0027, Japan

(Received 14 May 2001; revised manuscript received 8 October 2001; published 16 January 2002)

The scale-similar multifractal cascade, which is believed to describe the process of energy cascade in the inertial scale-range of isotropic turbulence, is proved to be Markovian so as to be governed by the Chapman-Kolmogorov equation, when negative logarithmic length scale is taken for time. For a limited class of cascades, the Kramers-Moyal expansion of the Chapman-Kolmogorov equation is possible and the coefficients are exactly derived from the functional form of intermittency exponents $\mu(q)$, and they are all constant if the cascade is scale similar.

DOI: 10.1103/PhysRevE.65.027301 PACS number(s): 47.27.Gs, 47.27.Jv, 47.53.+n, 02.50.−r

As is well known, many models have been proposed to explain the intermittency of velocity fluctuation in isotropic turbulence since the well-famed refined similarity hypothesis of Kolmogorov [1]. Most of them have the scale similarity that was clearly defined in Monin and Yaglom's book [2]; when ε_r is energy dissipation averaged over a domain of scale r, the probability distribution of $e_{r,l} \equiv \varepsilon_r/\varepsilon_l$ depends only on the ratio r/l, and moreover $l > \rho > r$ are three lengths in the inertial subrange, then the variables $e_{r,\rho}$ and $e_{\rho,l}$ are statistically independent. Here the domain of a larger scale is supposed to include that of a smaller scale. A necessary result from this is the existence of intermittency exponents $\mu(q)$ such that

$$\langle e_{r,l}^q \rangle = (r/l)^{-\mu(q)}, \tag{1}$$

where the angular bracket means the ensemble average. Typical examples of scale-similar intermittency models were discussed in [3,4], all of which are accompanied with their own, multifractal dissipation measures. The lognormal model of Kolmogorov [1] is also scale similar, as was described in [2].

Statistical independence of event $e_{r,l}$ seems to imply some Markovian nature in respect to length scale. [Indeed, the infinite divisibility of the probability density function (PDF) of $\ln e_{r,l}$, which was proved in [4] for scale-similar measures, is likely to suggest an interrelation with this nature.] To clarify this fact, we start from the general formula of the PDF of $\ln e_{r,l} \equiv \Delta x$.

First, there must be a characteristic function of Δx of the form

$$\phi(\theta; r/l) = \langle \exp[i\theta \ln e_{r,l}] \rangle = (r/l)^{\alpha(\theta)}. \tag{2}$$

[See [2]), $\alpha(\theta)$ in which should be $-\mu(i\theta)$ as is evident by comparison with Eq. (1)]. Hence we have the PDF of Δx as

$$p(\Delta x; r/l) = \int e^{-i\theta\Delta x} \phi(\theta; r/l)\, d\theta/2\pi = \int \exp[-i\theta\Delta x + \alpha(\theta)\ln(r/l)]\, d\theta/2\pi. \tag{3}$$

This form may be changed, by introducing a new parameter $\Delta t = -\ln r/l$, to

$$p(\Delta x; \Delta t) = \int \exp[-i\theta\Delta x - \alpha(\theta)\Delta t]\, d\theta/2\pi, \tag{4}$$

which can be interpreted as the conditional PDF of observing $x_2 = x_1 + \Delta x$ at the time $t_2 = t_1 + \Delta t$ when we observed x_1 at t_1. Based on Eq. (4), it is easy to derive the relation

$$p(x_2 - x_1; t_2 - t_1) = \int p(x_2 - x_3; t_2 - t_3) \times p(x_3 - x_1; t_3 - t_1)\, dx_3. \tag{5}$$

This is nothing but the Chapman-Kolmogorov equation restricted to a stationary and homogeneous case of the stochastic process. In this case, we can conveniently fix the origins of x and t by introducing the integral scale L and the average dissipation ε_L so that, for example, $x_2 = \ln e_{r,L}$ and $t_2 = -\ln r/L$.

The corresponding Kramers-Moyal expansion is readily obtained by differentiation of Eq. (4) as follows, *only if p is analytic in x_2*,

$$\begin{aligned}\partial p/\partial t_2 &= -\int \alpha(\theta)\exp[-i\theta\Delta x - \alpha(\theta)\Delta t]\, d\theta/2\pi \\ &= -\sum \alpha^{(n)}(0)/n! \int \theta^n \exp[-i\theta\Delta x - \alpha(\theta)\Delta t]\, d\theta/2\pi \\ &= -\sum \alpha^{(n)}(0)/[n!(-i)^n]\,\partial^n/\partial x_2^n p \\ &= \sum (-1)^n \mu^{(n)}(0)/n!\,\partial^n/\partial x_2^n p,\end{aligned} \tag{6}$$

keeping in mind the equality of $\alpha(\theta)$ and $-\mu(i\theta)$ noted before. Here we have used the analyticity of $\alpha(\theta)$ which comes from that of $\mu(q)$ [4]. Since $\mu(0) = 0$ (the condition of space-filling measure for isotropic turbulence, see [4]), n in the sum of Eq. (6) starts from $n = 1$. [Note that any nonspace-filling measure as in the β model cannot yield a Kramers-Moyal expansion, while it can satisfy the Chapman-Kolmogorov Eq. (5).] Thus, the expansion coefficients, say D_n, should be constant; that is $D_n = \mu^{(n)}(0)/n!$

as is clear from Eq. (6). This is an unveiled fact. For every scale-similar model which has a $p(\Delta x;\Delta t)$ analytic in Δx, all the coefficients can be calculated from the form of its proper $\mu(q)$. D_n are generally nonvanishing for n higher than 2. The only exception is the lognormal model, in which we have $\mu(q)=\mu q(q-1)/2$ $[\mu=\mu(2)]$ so that $D_1=-\mu/2$, $D_2=\mu/2$, and $D_n=0$ for all the higher n values, and then we have a simple Fokker-Planck equation from Eq. (6); the solution of which is

$$p(\Delta x;\Delta t)=\exp[-(\Delta x-D_1\Delta t)^2/(4D_2\Delta t)]/(4\pi D_2\Delta t)^{1/2}. \tag{7}$$

This is nothing but a Brownian motion in x space with a constant-speed drift.

Here should it be remarked that *all scale-similar models are Markovian in the sense that they can satisfy Eq. (5) but not all of them are expansible in the form of Eq. (6)*. Many good models even with space-filling measures are not expansible if their $p(\Delta x;\Delta t)$ are not analytic but singular or discontinuous functions. The p model [5] and the three-dimensional Cantor set model [3] are just such cases, as is known from the formulas shown in [3]. $D_n=\mu^{(n)}(0)/n!$ can be calculated for these models, but they are meaningless as the Kramers-Moyal expansion coefficients. In other words, the Kramers-Moyal expansion is not essential for $p(\Delta x;\Delta t)$ in a scale-similar model.

As an example of calculating D_n, we take up the log-Poisson model [6] with $\mu(q)=2q/3-2[1-(2/3)^q]$, assuming that $p(\Delta x;\Delta t)$ for this case is analytic (since its analyticity is not yet evident). The D_n values can be calculated, rather easily with the aid of a computer. The result is $D_1=-0.1443$, $D_2=0.1644$, $D_3=-0.0222$, $D_4=2.25\times10^{-3}$, $D_5=-1.83\times10^{-4}$, $D_6=1.23\times10^{-5}$, $D_7=-7.15\times10^{-7}$, $D_8=3.62\times10^{-8}$, $D_9=-1.63\times10^{-9}$, $D_{10}=6.62\times10^{-11}$ and the like. The magnitudes of the expansion coefficients decrease very rapidly with order n.

This may tempt us to think that the lognormal model is almost a sufficient approximation to another model. This is right so long as we look at the main part of the PDF alone. But once we look into the high moments of $e_{r,L}=\exp[x_2]$ (not x_2!) to which the intermittency in real turbulence greatly contributes, the approximation turns out to become poor, as has been well known by the fact that the lognormal model cannot explain the observed scalings of energy dissipation as well as velocity increment [7]. Therefore all the coefficients, if the Kramers-Moyal expansion is possible, should be non-negligible at all in principle, so far as scale-similarity holds. *These coefficients actually characterize the Markov process built in a scale-similar model which has an analytic* $p(\Delta x;\Delta t)$.

The recent work of Naert, Friedrich, and Peinke [8] proposes another type of Fokker-Planck equation for $p(\Delta x;\Delta t)$, which has a nonconstant D_1, based on their experimental data. It is a remarkable fact that they directly verified by experiment that the cascading process was Markovian. From our argument developed above, however, it is clear that their PDF of x must generally lack the scale similarity, because all D_n should be constant if the cascade is scale similar and if the expansion is possible. At this point their Fokker-Planck equation is a significant challenge to the entire concept of scale similarity, if it is universal for higher Reynolds numbers. Only in the limit when their D_1 goes to a constant ($\gamma\to0$ in [8]), their PDF of x is in accord with Eq. (7) to recover the scale similarity exactly as the lognormal model at the most. So, it looks important at present to study further on how universal their Markov process is.

[1] A. N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962).
[2] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (The MIT Press, Cambridge, 1975).
[3] I. Hosokawa, Phys. Rev. Lett. **66**, 1054 (1991).
[4] I. Hosokawa, Proc. R. Soc. London, Ser. A **453**, 691 (1997).
[5] C. Meneveau and K. R. Sreenivasan, Phys. Rev. Lett. **59**, 1424 (1987).
[6] Z.-S. She and E. Leveque, Phys. Rev. Lett. **72**, 336 (1994); it was proven in [4] that their model is a kind of (infinitely) multinomial Cantor set.
[7] U. Frisch, *Turbulence—The Legacy of Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[8] A. Naert, R. Friedrich, and J. Peinke, Phys. Rev. E **56**, 6719 (1997).